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## Exact field-theoretical description of passive scalar convection in an $N$ -dimensional long-range velocity field

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### Abstract

We describe a new functional integral method for the computation of averages containing chronological exponentials of random matrices of arbitrary dimension. We apply these results to the rigorous study of the statistics of a passive scalar advected by a large-scale  $N$ -dimensional flow. In the delta-correlated case the statistics of the rate of line stretching appears to be exactly Gaussian at all times and we explicitly compute the dependence of the mean value and variance of the stretching rate on the space dimension  $N$ . The probability distribution function of the passive scalar is also exactly computed. Further applications of our functional integral method are suggested.

### 1. Introduction

Many exact results, explaining the advection of a passive scalar by an  $N$ -dimensional large-scale flow, have been known in turbulence theory from the very beginning. Batchelor [1] found exactly the form of the double passive scalar correlator in the case of a slow external velocity field. Kraichnan [2] established the Gaussian character of the statistics of the line element in an  $N$ -dimensional flow in the limit of a velocity field  $\delta$ -correlated in time. For the same problem Cocke [3] proved the positivity of the rate of line stretching for a velocity field arbitrarily correlated in time. Recently, Shraiman and Siggia [4] found that at values exceeding the variance the passive scalar probability distribution function (p.d.f.) is exponential, at least for a shortly correlated velocity field. The two-dimensional problem was considered in detail in Ref. [5], where a central limit theorem for the statistics of the stretching rate was proven for a velocity field arbitrarily correlated in time and the rate of line stretching was calculated analytically. It was also proven there that, whatever the statistics of the velocity field be, the statistics of the passive scalar (averaged over time locally in space) approaches Gaussianity with the increase of the Peclet number  $Pe$  (the pumping-to-diffusion scale ratio).

The present paper is devoted to the generalization to arbitrary space dimensions of the field-theoretical approach proposed for the 2D case in Ref. [5].

As it was shown by Kraichnan [2] and afterwards slightly reformulated by Falkovich and Lebedev [6] (see Section 2, where we follow the formulation of Falkovich and Lebedev), the statistics of a passive scalar advected by a random large-scale flow is completely described by the statistics of the line element in the flow. Thus, to study the statistics of the passive scalar we have to find the statistics of the line stretching first.

In Section 3 (see also Appendix A) we describe a method for the computation of averages of functionals which contain a time-ordered exponential of an  $N \times N$  traceless Hermitian matrix function. This method is a generalization of the one firstly introduced by Kolokolov [7,8] in the  $N = 2$  case in the context of the theory of ferromagnets, and afterwards [9] applied to the study of one-dimensional localization.

In the general  $N$ -dimensional case we establish a qualitative picture which is similar to the one observed in Ref. [5] for the 2D problem. The separation  $R$  of two points of a passive scalar blob embedded in the velocity flow stretches exponentially. The statistics of the rate of line stretching appears to be Gaussian with nonzero mean value  $\bar{\lambda}$  and variance  $\Delta$ . The dependence of  $\bar{\lambda}$  and  $\Delta$  on the dimension  $N$  is calculated explicitly in the case of a shortly correlated flow when the stretching rate p.d.f. is shown to be exactly Gaussian at all times.

Finally, in Section 4 we find the local (without spatial averaging) p.d.f. of the passive scalar  $\theta(\mathbf{r})$  and the p.d.f. of the simultaneous product  $\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)$  in the case of a shortly correlated flow explicitly, through the computation of certain functional integrals of an auxiliary quantum mechanical problem.

## 2. Formulation of the problem

The advection of the scalar field  $\theta(t, \mathbf{r})$  is governed by the following equation,

$$\dot{\theta} + u_\alpha \nabla_\alpha \theta = \phi, \quad (1)$$

where  $\mathbf{u}(t, \mathbf{r})$  is the external velocity field and  $\phi(t, \mathbf{r})$  is an external source, which are both assumed to be random functions of  $t$  and  $\mathbf{r}$ . We suppose that the source  $\phi$  be correlated on a scale  $L$  and arbitrarily correlated in time. This means e.g. that the pair correlation function of the source  $\langle \phi(\mathbf{r}_1, t_1) \phi(\mathbf{r}_2, t_2) \rangle = \Xi(t_1 - t_2, \mathbf{r}_{12})$ , as a function of the argument  $\mathbf{r}_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$ , decays on the scale  $L$ . The same behavior is assumed for high-order correlation functions of the source. The velocity field may be multi-scale, its smallest scale being assumed to be larger than or of the order of  $L$ . In the following we will specialize to the case of  $\langle \phi(\mathbf{r}_1, t_1) \phi(\mathbf{r}_2, t_2) \rangle = P_2 \xi_2(\mathbf{r}_{12}) \delta(t_1 - t_2)$ , where  $P_2$  has the physical meaning of production rate of  $\theta^2$ . For simplicity we can simply put  $\xi_2(x) = 1$  for  $x < L$  and  $\xi_2(x) = 0$  for  $x > L$  (a more precise account of the form of  $\xi_2$  will not change the results up to logarithmic accuracy).

Falkovich and Lebedev [6] showed that in the problem of finding the steady statistics of a passive scalar advected by a large-scale velocity field the averaging with respect to the velocity field and with respect to the external source can be separated. Such an approach was initially applied to the 2D case, but it maintains its validity in any space dimension. We will here formulate the problem following the same approach but omitting the details, which can be found in Refs. [6,5]. We have then

$$\theta(t, \mathbf{r}) = \int_0^{+\infty} dt' \phi(t - t', \hat{W}(t) \mathbf{r}), \quad (2)$$

where  $\hat{W}(t)$  satisfies  $\dot{\hat{W}} = \hat{\sigma} \hat{W}$ ,  $\hat{W}(0) = \mathbf{1}$  and  $\sigma^{\alpha\beta} \equiv \nabla^\alpha v^\beta(0, t)$  is the  $N \times N$  matrix of the derivatives of the quasi-Lagrangian velocities  $v_\beta$  referred to a frame which is locally comoving with the fluid at the point  $\mathbf{r} = 0$ .  $\hat{\sigma}$  is assumed to be a traceless symmetric matrix random in time.

The computation of the statistics of the simultaneous product  $\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)$  is thus reduced to the averaging of

$$Q = P_2 \int_0^\infty \xi_2(R(t)) dt, \quad (3)$$

where now  $\mathbf{R}(t) = \dot{\mathbf{W}}(t) \cdot (\mathbf{r}_1 - \mathbf{r}_2)$  describes the separation of two fluid particles. This way the problem of finding the passive scalar statistics has been transformed to a purely kinematical one. Alternatively,  $\lambda(t) = \ln[R(t)/r]/t$  may be considered as the rate of line stretching along some axis (whose particular choice is inessential since the velocity field statistics was chosen to be isotropic) of a cloud embedded into the velocity field. Let us note that the kinematical problem of studying the dispersion of a pair of  $N$ -dimensional Lagrangian tracers is by itself of great interest.

The matrix  $\hat{\sigma}$  describes the local structure of the flow in the vicinity of the point  $\mathbf{r} = 0$ . We can analyze it by decomposing  $\hat{\sigma}$  in the standard way,

$$\hat{\sigma} = \hat{\mathcal{R}} + \hat{\mathcal{S}}_l + \hat{\mathcal{S}}_h, \quad \hat{\mathcal{R}} = \begin{pmatrix} 0 & c_{1,2} & \dots & c_{1,N} \\ c_{2,1} & 0 & & \vdots \\ \vdots & & \ddots & \\ c_{N,1} & \dots & c_{N,N-1} & 0 \end{pmatrix}, \quad c_{i,j} = -c_{j,i}, \quad (4)$$

$$\hat{\mathcal{S}}_l = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & 0 & a_N \end{pmatrix}, \quad a_N = -\sum_{i=1}^{N-1} a_i, \quad \hat{\mathcal{S}}_h = \begin{pmatrix} 0 & b_{1,2} & \dots & b_{1,N} \\ b_{2,1} & 0 & & \vdots \\ \vdots & & \ddots & \\ b_{N,1} & \dots & b_{N,N-1} & 0 \end{pmatrix}, \quad b_{i,j} = b_{j,i}. \quad (5)$$

Here,  $\hat{\mathcal{R}}$  is the antisymmetric part of  $\hat{\sigma}$ , inducing a rotation of the passive scalar blob. The symmetric part  $\hat{\mathcal{S}}$  can be split in two:  $\hat{\mathcal{S}} = \hat{\mathcal{S}}_l + \hat{\mathcal{S}}_h$ ,  $\hat{\mathcal{S}}_l$  representing the stretching of the unit blob under the longitudinal velocity gradient and  $\hat{\mathcal{S}}_h$  representing the shearing under the transverse velocity gradient. We will restrict to the case of an incompressible fluid, so that  $\hat{\mathcal{S}}_l$  will be assumed to be traceless. We assume space isotropy of the strain so that  $\hat{\mathcal{R}}$ ,  $\hat{\mathcal{S}}_l$  and  $\hat{\mathcal{S}}_h$  are independent matrix processes random in time with zero mean and  $\langle \text{Tr}[\hat{\mathcal{S}}_l^2] \rangle = \langle \text{Tr}[\hat{\mathcal{S}}_h^2] \rangle$ . In this paper we will restrict ourselves mainly to the particular case of a statistics of  $\hat{\sigma}$  delta-correlated in time,

$$\mathcal{DM}[\hat{\sigma}] = \mathcal{D}\hat{\sigma} \exp\left(-\frac{1}{2D_s} \int \mathcal{L} dt\right), \quad \mathcal{L} = \frac{1}{2} \left( \text{Tr}[\hat{\mathcal{S}}^2] - \frac{D_s}{D_r} \text{Tr}[\hat{\mathcal{R}}^2] \right). \quad (6)$$

We introduce two *a priori* different values  $D_s$  and  $D_r$ , characterizing the strain and vorticity amplitudes. Since all the averages we want to compute contain only the modulus  $R = |\mathbf{R}|$  we can split  $\dot{\mathbf{W}}$  into the product  $\dot{\mathbf{W}} = \dot{\mathbf{W}}_r \dot{\mathbf{W}}_s$ , whose factors satisfy the following separate equations,

$$\dot{\mathbf{W}}_r + \hat{\mathcal{R}} \dot{\mathbf{W}}_r = 0, \quad \dot{\mathbf{W}}_s + \hat{\mathcal{S}}' \dot{\mathbf{W}}_s = 0, \quad (7)$$

where  $\hat{\mathcal{S}}' = \dot{\mathbf{W}}_r^{-1} \hat{\mathcal{S}} \dot{\mathbf{W}}_r$ . Since  $\dot{\mathbf{W}}_r^T = \dot{\mathbf{W}}_r^{-1}$  we have  $|\dot{\mathbf{W}}_r \mathbf{R}| = R$ , and since the strain part of the measure (6) is invariant with respect to the  $\hat{\mathcal{S}} \rightarrow \hat{\mathcal{S}}'$  transformation (which is generally speaking not true in the finitely correlated case) we can completely exclude vorticity from our considerations and we have the freedom to choose  $D_r$  arbitrarily. This will be useful for the computations of Section 3.

### 3. Averages of time-ordered exponentials

In this section we will consider the problem of computing averages of the form

$$\langle f(\hat{W}) \rangle = \int \mathcal{D}\hat{\eta} \exp \left( -\frac{1}{4D} \int_0^T \text{Tr} \hat{\eta}^2 \right) f(\hat{W}(T)), \quad (8)$$

where  $\hat{\eta}(t)$ , for  $0 \leq t \leq T$ , is a Hermitian traceless  $N \times N$  matrix function and

$$\hat{W}(t) = T \exp \left( -\int_0^t \hat{\eta}(s) ds \right) \quad (9)$$

is an  $SL(N)$  matrix function, solution of  $\dot{\hat{W}} = \hat{\eta} \hat{W}$  with  $\hat{W}(0) = \mathbf{1}$ , given here in the form of a time-ordered exponential. The average (8) is difficult to compute due to the complicated form of (9). We will now introduce a non-linear variable change which reduces (9) to a product of usual matrix exponentials without spoiling the Gaussian character of the averaging weight. This method was firstly introduced by Kolokolov [7] in order to deal with the  $SL(2)$  case, which is relevant in the theory of the quantum Heisenberg ferromagnet. Here we will describe the generalization of this method to the  $SL(N)$  case.

Let us start by Gauss-decomposing the matrix  $\hat{W}(t)$ <sup>1</sup>,

$$\begin{aligned} \hat{W}(t) = & \begin{pmatrix} 1 & 0 & \dots & 0 \\ \psi_{2,1}^-(t) & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ \psi_{N,1}^-(t) & \dots & \psi_{N,N-1}^-(t) & 1 \end{pmatrix} \times \begin{pmatrix} \exp \left( \int_0^t \rho_1 d\tau \right) & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & & 0 \\ 0 & \dots & 0 & \exp \left( \int_0^t \rho_N d\tau \right) \end{pmatrix} \\ & \times \begin{pmatrix} 1 & F_{1,2}(t) & \dots & F_{1,N}(t) \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & \dots & 0 \\ \psi_{2,1}^-(0) & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ \psi_{N,1}^-(0) & \dots & \psi_{N,N-1}^-(0) & 1 \end{pmatrix}^{-1}, \\ \rho_N(t) = & -\sum_{i=1}^{N-1} \rho_i(t). \end{aligned} \quad (10)$$

After computing  $\hat{\eta} = \dot{\hat{W}} \hat{W}^{-1}$  it is easy to show that

$$\frac{1}{2} \text{Tr}(\hat{\eta}^2) = \frac{1}{2} \sum_i \rho_i^2 + \sum_{i,j=1; i>j}^N \psi_{i,j}^+(\rho, F, \dot{F}, \psi^-) \psi_{i,j}^-, \quad (11)$$

<sup>1</sup> Note that the form of (10), together with the initial condition  $\hat{W}(0) = \mathbf{1}$ , imply  $F_{i,j}(0) = 0$ , while the initial values  $\psi_{i,j}^-(0)$  are kept arbitrary.

where the  $\psi_{i,j}^{\pm}$  are certain rational functions of the arguments. We will then take the fields  $\psi_{i,j}^{\pm}$  as new integration variables. As a matter of fact, it appears that the elements  $\eta_{i,j}$  can be re-expressed as differential polynomials in the new variables  $(\rho_i, \psi_{i,j}^{\pm})$  (see in Appendix A the  $N = 3$  case as an example). After such a variable transformation the averaging weight (11) maintains by definition its Gaussian form. Of course we have yet to compute the Jacobian of the transformation  $(\eta_{i,j}) \rightarrow (\rho_i, \psi_{i,j}^{\pm})$ , which could in principle give a non-linear contribution. Fortunately this is not the case and the Jacobian has the following simple form (see Appendix A),

$$\mathcal{J}_N \propto \exp \left( \sum_{i=1}^{N-1} (N-i) \int_0^T \rho_i(t) dt \right). \quad (12)$$

Let us apply these results to the computation of the statistics of the stretching rate. Exploiting the isotropy condition we can choose  $R^a(0) = r \delta_{1a}$  as the initial value of  $R$ . From (10) we thus get

$$R(t) = r \left( 1 + \sum_{i=2}^N (\psi_{i1}^-)^2 \right)^{1/2} \exp \left( \int_0^t \rho_1(\tau) d\tau \right). \quad (13)$$

This immediately gives the desired asymptotic expression for the stretching rate,

$$\lambda(T) \simeq \frac{1}{T} \int_0^T \rho_1(\tau) d\tau, \quad T \rightarrow \infty. \quad (14)$$

Up to this point we have not made use of any particular property of the measure (6). We thus succeeded in finding a representation of the stretching rate as an integral of a scalar function for arbitrary statistics of the velocity field. If we could show that  $\rho_1$  is a field finitely correlated in time we would immediately get a proof of a central limit theorem for the statistics of  $\lambda(T)$ . This program can be carried on thoroughly in the same way as it was done for the  $N = 2$  case in Ref. [5]. It is worth noticing that the positivity of  $\lambda$ , firstly established by Cocke [3], is essential in proving the “clusterization” property of the field  $\rho_1$ . Note lastly that still another way of proving a central limit theory for the process has been recently proposed in Ref. [13].

In the case of a delta-correlated statistics of the velocity derivatives we are able to compute the statistics of  $\lambda$  explicitly at arbitrary times. From (6) we have

$$\mathcal{L} = \frac{1}{2} \text{Tr}[(\hat{S} + i\alpha \hat{R})^2], \quad (15)$$

where  $\alpha = \sqrt{D_s/D_r}$ . However, we observed in Section 2 that the statistics of  $\lambda$  cannot depend on  $\alpha$ . Let us make use of this arbitrariness by substituting  $\alpha \rightarrow -i$ ,  $R \rightarrow iR$ , and therefore getting  $\hat{\sigma} \rightarrow \hat{\eta} = \hat{S} + i\hat{R}$ ,  $\mathcal{L} \rightarrow \frac{1}{2} \text{Tr}(\hat{\eta}^2)$ ,  $\mathcal{D}\hat{\sigma} \rightarrow \mathcal{D}\hat{\eta}$ . We thus reduce to the already studied problem (9). After the variable transformation  $(\eta_{i,j}) \rightarrow (\rho_i, \psi_{i,j}^{\pm})$  we get

$$\mathcal{DM} = \mathcal{D}\rho \mathcal{D}\psi^{\pm} \exp \left( -\frac{1}{2D_s} \int_0^T \tilde{\mathcal{L}} dt \right), \quad \tilde{\mathcal{L}} = \frac{1}{2} \sum_{i=1}^N \rho_i^2 + \sum_{i,j=1; i>j}^N \dot{\psi}^- \psi^+ - 2D_s \sum_{i=1}^{N-1} (N-i) \rho_i. \quad (16)$$

The effective action (16) contains only terms which are “neutral” with respect to the  $\pm$  “charge”. Thus we immediately recognize that in the delta-correlated case expression (14) is exact at arbitrary times: the “non-neutral” terms  $(\psi^-)^2$  give a zero contribution and (13) can be replaced by

$$R(t) = \exp \left( \int_0^t \rho_1(\tau) d\tau \right). \quad (17)$$

Performing Gaussian integration with respect to  $\psi^\pm$  and  $\rho_2, \dots, \rho_N$  we obtain the following exact results for the statistics of the field  $\rho_1$ ,

$$\mathcal{D}\rho_1 \exp\left(-\frac{N}{4D_s(N-1)} \int_0^T [\rho_1 - D_s(N-1)]^2 d\tau\right) \quad (18)$$

and for the p.d.f. of the stretching rate  $\lambda$ ,

$$\mathcal{P}[\lambda; T] = \frac{1}{2} \sqrt{\frac{NT}{\pi D_s(N-1)}} \exp\left(-\frac{NT}{4D_s(N-1)} [\lambda - D_s(N-1)]^2\right); \quad (19)$$

the latter appears to be exactly Gaussian with Lyapunov exponent  $\lambda = D_s(N-1)$  and variance  $\Delta = 2D_s(N-1)/NT$ . For  $T$  tending to infinity  $\lambda$  tends asymptotically to a deterministic quantity. The ratio  $2\lambda/\Delta$ , in accord with Kraichnan [2], is proportional to the space dimension  $N$ .

#### 4. Passive scalar statistics

A point of great physical interest is the computation of the p.d.f. of the passive scalar. Carrying over to arbitrary dimensions the approach shown in Ref. [5] we observe that all the passive scalar correlators can be calculated directly from the p.d.f. of the stretching rate. Here we will limit ourselves to the computation of the p.d.f. of the simultaneous product  $Q$  and to the single point p.d.f. of the passive scalar. For this purpose we reduce the problem to an auxiliary exactly solvable quantum mechanical problem (see Appendix B). In such a way we are able to obtain explicitly (see (B.4)) the Laplace transform

$$\mathcal{P}_L(s) = \lim_{T \rightarrow \infty} \langle \exp(-sQ_T) \rangle, \quad Q_T = P_2 \int_0^T \xi_2(R(t)) dt. \quad (20)$$

$\mathcal{P}_L$  is defined on the complex plane with a cut going from  $-N(N-1)D_s/4P_2$  to  $-\infty$ . Thus,

$$\begin{aligned} \mathcal{P}(Q) &= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{sQ} \mathcal{P}_L(s) ds = \frac{N(N-1)D_s}{4P_2} \mathcal{F}(QD/2P_2), \\ \mathcal{F}(y) &= \frac{2}{\pi} \exp[-y + \ln(L/r)] \int_{-\infty}^{+\infty} \frac{x dx}{1-ix} \exp[-yx^2 + ix \ln(L/r)]. \end{aligned} \quad (21)$$

For  $y \equiv QN(N-1)D_s/4P_2 \sim \ln(L/r) \gg 1$  the integral (21), computed by means of a saddle-point approximation, gives

$$\mathcal{F}(y) \approx \frac{2}{\sqrt{\pi y}} \frac{\ln(L/r)}{2y + \ln(L/r)} \exp\left(-\frac{1}{4y} [2y - \ln(L/r)]^2\right), \quad (22)$$

and shows a Gaussian bump with a non-Gaussian tail. The predexponential factor in (22) is correct for finite deviations of  $y$  from the mean value (within many dispersion intervals) as long as  $y \gg 1$ .

The exact result (B.4) for  $\mathcal{P}_L(s)$  can also be used to compute the single-point p.d.f. of the passive scalar  $\mathcal{P}_{sp}(w)$ . Indeed, the higher order correlation functions  $\langle \theta^{2n} \rangle$  differ from  $\langle Q^n(r_{dif}) \rangle$  only for a combinatorial factor

$$\langle \theta^{2n} \rangle = (2n-1)!! \langle Q^n(r_{dif}) \rangle, \quad (23)$$

where we replaced the argument of  $Q$  with the diffusion scale and correspondingly  $r/L$  with the Peclet number  $Pe \equiv r_{\text{dif}}/L$  (the pumping-to-diffusion scale ratio). Thus, for the Laplace transform of the p.d.f. of  $\theta$  we get

$$\langle e^{-s\theta} \rangle = \sum_{n=0}^{\infty} \frac{s^{2n} \langle \theta^{2n} \rangle}{(2n)!} = \sum_{n=0}^{\infty} \frac{s^{2n} \langle Q^n \rangle}{(2n)!!} = \langle \exp(\frac{1}{2}s^2 Q) \rangle = \mathcal{P}_L(-\frac{1}{2}s^2). \quad (24)$$

This gives the following exact expression for the single-point p.d.f. of  $\theta$ ,

$$\mathcal{P}_{\text{sp}}(\theta) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{sQ} \mathcal{P}_L(-s^2/2) ds = \frac{1}{2\pi} \sqrt{\frac{N(N-1)D_s}{2P_2}} \int_{-\infty}^{+\infty} e^{izq} \frac{\exp[\ln(L/r_{\text{dif}})(1 - \sqrt{1+q^2})]}{1 + \sqrt{1+q^2}} dq, \quad (25)$$

where  $z = \theta \sqrt{N(N-1)D_s/2P_2}$ . After a saddle-point evaluation of the integral (25) we get the following approximation for  $\mathcal{P}_{\text{sp}}(\theta)$  for  $\theta \sqrt{N(N-1)D_s/2P_2} \sim \ln(L/r_{\text{dif}}) \gg 1$ ,

$$\mathcal{P}_{\text{sp}}(\theta) \propto \exp\left(-\sqrt{\theta^2 N(N-1)D_s/2P_2 + \ln^2(L/r_{\text{dif}})}\right). \quad (26)$$

This formula shows the presence of an exponential tail. The exponent does not depend on the Peclet number. Thus, our exact result (25), generalizing the already obtained two-dimensional results [5], supports the basic statement about the exponential tail of the single-point p.d.f. of  $\theta$  advanced by Shraiman and Siggia [4].

## 5. Conclusions

The formalism developed in Section 2 can be applied to many other problems: let us mention in the first place the study of  $N$ -level quantum-mechanical systems affected by a random noise of general kind. This problem can be reduced to the computation of averages of a time-ordered exponential of anti-Hermitian random matrices. In Ref. [14] the averaged density matrix for this problem was computed by a direct expansion, and essential use was made of algebraic properties which would not hold in the Hermitian case. With our method we can compute all the probability distribution functions directly. Moreover in the  $N = 2$  case it has been shown [5] that our formalism can be adapted to the case of a finitely-correlated noise (“coloured” noise). There are no principal obstacles to the generalization of this approach to arbitrary  $N$ , and it would be interesting to study by this method the statistics of a passive scalar (and more generally of a passive tensor) advected by a random velocity field with finite correlation time in 3D space.

Our formalism also provides an alternative to the supersymmetric approach [15,16] to the problem of  $N/2$  channels localization (see Ref. [9] for the  $N = 2$  case).

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## Appendix A. Computation of the Jacobian

Let us show the explicit form of the variable transformation

$$\eta_{i,j} = \eta_{i,j}(\rho, \psi^{\pm}) \quad (A.1)$$

in the  $N = 3$  case:

$$\begin{aligned}
 \eta_{11} &\equiv a_1 = \rho_1 - \psi_{2,1}^+ \psi_{2,1}^- - \psi_{3,1}^+ \psi_{3,1}^-, & \eta_{22} &\equiv a_2 = \rho_2 + \psi_{2,1}^+ \psi_{2,1}^- - \psi_{3,2}^+ \psi_{3,2}^-, \\
 \eta_{1,2} &\equiv b_{1,2} + ic_{1,2} = \psi_{2,1}^+, & \eta_{1,3} &\equiv b_{1,3} + ic_{1,3} = \psi_{3,1}^+, & \eta_{2,3} &\equiv b_{2,3} + ic_{2,3} = \psi_{3,2}^+ + \psi_{2,1}^- \psi_{3,1}^+, \\
 \eta_{2,1} &\equiv b_{1,2} - ic_{1,2} = \dot{\psi}_{2,1}^- + \psi_{2,1}^- (\rho_1 - \rho_2) - \psi_{2,1}^+ (\psi_{2,1}^-)^2 - \psi_{3,1}^+ (\psi_{2,1}^- \psi_{3,1}^-) + \psi_{3,2}^+ (\psi_{2,1}^- \psi_{3,2}^- - \psi_{3,1}^-), \\
 \eta_{3,2} &\equiv b_{2,3} - ic_{2,3} = \dot{\psi}_{3,2}^- + (\rho_1 + 2\rho_2) \psi_{3,2}^- + \psi_{3,1}^- \psi_{2,1}^+ - \psi_{3,2}^+ (\psi_{3,2}^-)^2, \\
 \eta_{3,1} &\equiv b_{1,3} - ic_{1,3} = \dot{\psi}_{3,1}^- - \psi_{2,1}^- \dot{\psi}_{3,2}^- + \rho_1 (2\psi_{3,1}^- - \psi_{2,1}^- \psi_{3,2}^-) + \rho_2 (\psi_{3,1}^- - 2\psi_{2,1}^- \psi_{3,2}^-) - \psi_{2,1}^+ \psi_{2,1}^- \psi_{3,1}^+ \\
 &\quad - \psi_{3,1}^+ (\psi_{3,1}^-)^2 + \psi_{3,2}^+ (\psi_{2,1}^- (\psi_{3,2}^-)^2 - \psi_{3,2}^- \psi_{3,1}^-).
 \end{aligned} \tag{A.2}$$

The form of (A.1) for  $N > 3$  can be easily obtained from the prescription (11). In order to compute the Jacobian of the transformation we will make use of the following properties of (A.1). First, observe that (11) is invariant with respect to the phase shift

$$\psi_{i,j}^\pm \rightarrow e^{\pm \alpha_{i,j}} \psi_{i,j}^\pm, \quad \sum_{i,j} \alpha_{i,j} (-1)^{i+j} = 0, \tag{A.3}$$

while it can be easily checked that the functions (A.1) are covariant under the same transformation. Therefore the Jacobian  $\mathcal{J}_N = \partial(\eta)/\partial(\rho, \psi^\pm)$  will be invariant under (A.3) and contain only “neutral” terms. Remember that we can arbitrarily choose the initial values  $\psi_{i,j}^-(0)$ . This freedom is equivalent to the freedom in the choice of the orientation of  $\mathbf{R}(0)$ . This fact, together with the isotropy condition, implies that the averaging measure must be invariant with respect to the uniform shift  $\psi_{i,j}^-(t) \rightarrow \psi_{i,j}^-(t) + \text{const}_{i,j}$ . This means that  $\mathcal{J}_N$  must not depend on the fields  $\psi^-$  (a more accurate inspection shows that it cannot even depend on  $\dot{\psi}^-$ ). Neutrality then implies that  $\mathcal{J}_N$  cannot depend on  $\psi^+$  either. Lastly, one can show by direct computation that

$$\eta_{i,j} = \dot{\psi}_{j,i}^- - (\rho_j - \rho_i) \psi_{j,i}^- + \dots \quad (i > j), \tag{A.4}$$

where  $\rho_N = -\sum_{i=1}^{N-1} \rho_i$  and the dots stand for terms which are non-linear in  $\psi^\pm$  and therefore give no contribution to  $\mathcal{J}_N$ . A standard regularization procedure ( $\psi_n^\pm = \psi^\pm(t_n)$ ;  $n = 1, \dots, M$ ;  $h = t/M \rightarrow 0$ ;  $t_n = t + hn$ ;  $M \rightarrow \infty$ ;  $\psi^-(t) \rightarrow \frac{1}{2}(\psi_n^- + \psi_{n-1}^-)$ ; see Ref. [8]) produces then the simple expression (12).

Our approach generalizes the results of Ref. [7] to the case  $N > 2$ . Recently, expressions similar to (A.2) have been obtained in the framework of conformal field theory [17]. However, the explicit form of the variables  $\psi^+$  and of the Jacobian  $\mathcal{J}$ , which are essential for any physical application of our method, were not computed there.

## Appendix B. Associated quantum mechanics

Substituting  $\rho_1 = \dot{\zeta}, \zeta(0) = 0$  in (17) and (20) we reduce the computation of  $\mathcal{P}_L(s)$  to the following auxiliary quantum mechanical problem,

$$\begin{aligned}
 \mathcal{P}_L(s) &= \exp[-\tfrac{1}{4}ND_s(N-1)t] \int_{\zeta(0)=0}^T \mathcal{D}\zeta \exp\left[-\int_0^T \left(\frac{N}{4D_s(N-1)}\dot{\zeta}^2 + P_2 s \zeta_2(e^{\zeta} r)\right) dt + \tfrac{1}{2}N\zeta(T)\right] \\
 &= \exp[-\tfrac{1}{4}ND_s(N-1)t] \langle \delta(\zeta) | e^{-\hat{H}T} | e^{N\zeta/2} \rangle,
 \end{aligned} \tag{B.1}$$

with Hamiltonian

$$\hat{H} = -\frac{D_s(N-1)}{N} \partial_\zeta^2 + P_2 s \zeta_2(r e^\zeta). \tag{B.2}$$



The last average in (B.1) designates a matrix element of  $\exp(-\hat{H}t)$  between states described by the corresponding wave functions. Let us take for  $\xi_2(x)$  the step function  $\theta(L-x)$ : this will give us the correct answer up to logarithmic corrections. Then  $\hat{H} = -[D_s(N-1)/N]\partial_\zeta^2 + U(\zeta)$ , where

$$U(\zeta) = U_s = P_2 s, \quad \zeta < \zeta_0 = \ln(L/r), \\ = 0, \quad \zeta > \zeta_0 \quad (\text{B.3})$$

Thus,

$$\mathcal{P}_L(s) = \exp[-\frac{1}{4}ND(N-1)t]\Psi(\zeta=0, t=0),$$

where  $\Psi$  is defined by the following initial value problem:  $\partial_t \Psi = -\hat{H}\Psi$ ,  $\Psi(\zeta, T) = e^{N\zeta/2}$ . For  $\zeta \rightarrow \infty$  the exponentially growing solution is proportional to  $\exp[\frac{1}{4}ND(N-1)(T-t) + \frac{1}{2}N\zeta]$  for any  $t$ . So, for  $\zeta \rightarrow +\infty$ ,  $\Psi(\zeta, t) \rightarrow e^{N\zeta/2}$ . Generally speaking  $\Psi(\zeta) \sim \exp[\frac{1}{4}ND_s(N-1)T]f(\zeta)$ , for  $t \rightarrow \infty$ , where  $f(\zeta)$  satisfies the equation

$$[\hat{H} + \frac{1}{4}ND_s(N-1)]f = 0$$

and has the asymptotic  $f(\zeta \rightarrow +\infty) = e^{N\zeta/2}$  and  $f(\zeta \rightarrow -\infty) < \infty$ . For the potential (B.3), this gives

$$f(\zeta) = e^{N\zeta/2} + A e^{-ND_s\zeta/2}, \quad \zeta > \zeta_0, \\ = B \exp\left(N\sqrt{1 + \frac{4P_2s}{N(N-1)D_s}}\zeta/2\right), \quad \zeta < \zeta_0.$$

Here, the constants  $A$  and  $B$  must be defined matching  $f$  and  $\partial_\zeta f$  at the point  $\zeta_0$ . We finally obtain

$$\mathcal{P}_L(s) = f(\zeta=0) = B \\ = \frac{2}{1 + \sqrt{1 + 4P_2s/N(N-1)D_s}} \exp\left(\ln \frac{L}{r} [1 - \sqrt{1 + 4P_2s/N(N-1)D_s}]\right). \quad (\text{B.4})$$

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